Probabilistic analysis of the number partitioning problem

1998 J. Phys. A: Math. Gen. 31 3417

(http://iopscience.iop.org/0305-4470/31/15/007)

J. Phys. A: Math. Gen. **31** (1998) 3417–3428. Printed in the UK PII: S0305-4470(98)89379-7

Probabilistic analysis of the number partitioning problem

F F Ferreira and J F Fontanari

Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-970 São Carlos SP, Brazil

Received 19 November 1997

Abstract. Given a sequence of N positive real numbers $\{a_1, a_2, \ldots, a_N\}$, the number partitioning problem consists of partitioning them into two sets such that the absolute value of the difference of the sums of a_j over the two sets is minimized. In the case in which the a_j's are statistically independent random variables uniformly distributed in the unit interval, this NP-complete problem is equivalent to the problem of finding the ground state of an infinite-range, random antiferromagnetic Ising model. We employ the annealed approximation to derive analytical lower bounds to the average value of the difference for the best constrained and unconstrained partitions in the large N limit. Furthermore, we calculate analytically the fraction of metastable states, i.e. states that are stable against all single spin flips, and found that it vanishes like $N^{-3/2}$.

1. Introduction

The importance of the study of complex optimization problems which involve quenched, random, frustrated functions of many variables, as well as the major role that statistical mechanics can play in the study, were pointed out by many authors more than 10 years ago [1, 2]. In fact, the well-established statistical mechanics techniques to characterize ground states (global minima) and metastable states (local minima) of spin glass models can be readily adapted to the study of optimization problems [2]. As a result, these techniques have been successfully applied to the probabilistic analysis of several classical combinatorial optimization problems, such as the graph partitioning problem [3], the travelling salesman problem [4, 5], the knapsack problem [6–8], and the satisfiability problem [9–11], to mention only a few.

In this paper we study the number partition problem (NPP) which is stated as follows. Given a sequence of positive real numbers $\{a_1, a_2, \ldots, a_N\}$, the NPP consists of partitioning them into two disjoint sets $\mathcal{A}_1$ and $\mathcal{A}_2$ such that the difference

$$\left| \sum_{a_j \in \mathcal{A}_1} a_j - \sum_{a_j \in \mathcal{A}_2} a_j \right| \tag{1}$$

is minimized. Alternatively, we can search for the Ising spin configurations $\boldsymbol{s} = (s_1, \ldots, s_N)$ that minimize the energy or cost function

$$E(\boldsymbol{s}) = \left| \sum_{j=1}^{N} a_j s_j \right| \tag{2}$$

0305-4470/98/153417+12$19.50

where $s_j = 1$ if $a_j \in \mathcal{A}_1$ and $s_j = -1$ if $a_j \in \mathcal{A}_2$. We can also consider the problem of constrained partitions, in which the difference between the cardinalities of sets $\mathcal{A}_1$ and $\mathcal{A}_2$ is fixed, i.e.

$$m = \frac{1}{N}\sum_{j=1}^{N} s_j. \tag{3}$$

The NPP may be viewed as the practical problem of finding the fairest way to partition a set of N objects $j = 1, 2, \ldots, N$, each of which of value a_j, between two people. Despite its simplicity, the NPP was shown to belong to the NP-complete class, which basically means that there is no known deterministic algorithm guaranteed to solve all instances of this problem within a polynomial time bound [12]. The fact that the NPP is frustrated can easily be understood by squaring equation (2), so that the problem of minimizing E becomes the one of finding the ground state of the infinite-range, random antiferromagnetic Ising Hamiltonian [13]

$$\mathcal{H} = \tfrac{1}{2}\sum_{i}\sum_{j>i} a_i a_j s_i s_j. \tag{4}$$

Thus, we note that the problem of finding the ground state of (4) is NP-complete.

Although zero-cost solutions of the NPP may be of some value to cryptography [14], the interest in this problem stems mainly from the remarkable failure of the stochastic heuristic simulated annealing [15, 16] to find good solutions to it, as compared with the solutions found by deterministic heuristics [17]. In fact, the reason for that failure is that the usual strategy of exploring the space of configurations $\{\boldsymbol{s}\}$ through single spin flips leads to changes of energy that are typically of order $1/N$, while a theoretical analysis indicates that the global minimum energy is of order $\sqrt{N}2^{-N}$ for unconstrained partitions [18]. It is interesting to note that a very simple deterministic heuristic, the differencing method of Karmakar and Karp [19], can find with high probability solutions whose energies are of order $1/N^{\alpha \log N}$ for some $\alpha > 0$. More recently, it has been shown that the performance of simulated annealing can be greatly improved and even surpass that of the differencing method by employing different representations for the problem [20].

In this work we employ the annealed approximation [21, 22] to derive rigorous lower bounds to the average value of the difference or energy for the best constrained and unconstrained partitions. For constrained partitions, we show that the average optimal energy is extensive for $m > \sqrt{2} - 1$ and we calculate it exactly in this regime using the self-averaging property of the free energy density. The theoretical predictions are compared with numerical estimates for the optimal energy obtained through the exhaustive search of the configuration space for $N \leqslant 24$. Furthermore, we calculate analytically the average number of minima in the 1-swap neighbourhood and estimate their typical energy. A minimum in the 1-swap neighbourhood is a state that has lower energy than all the N states that differ from it by a single spin only [17]. Similarly to previous studies of the NPP [17, 18, 20], we will consider the case where the a_j's are statistically independent random variables uniformly distributed in the unit interval.

The remainder of this paper is organized as follows. In section 2 we describe the annealed approximation and calculate the lower bounds to the average value of the optimal energy. In section 3 we present the calculation of the average number of local minima in the 1-swap neighbourhood. In section 4 we discuss our main results and present some concluding remarks. In particular, we compare our approach with other theoretical studies of the NPP [13, 18]. In the appendix we present the details of the self-averaging calculation of the average optimal energy in the regime where this quantity is extensive.

2. Annealed approximation

In the canonical ensemble formalism of the statistical mechanics the average value of the optimal energy for constrained partitions is given by

$$\bar{E}_m = \lim_{T\to 0} F_m(T) = -\lim_{T\to 0} T\langle \ln Z_m\rangle \tag{5}$$

where $F_m(T)$ is the average free energy, and $Z_m(T)$ is the partition function

$$Z_m(T) = \sum_{\{s\}} \delta\left(Nm, \sum_j s_j\right) \exp\left[-\frac{E(s)}{T}\right] \tag{6}$$

with $m = -1, -1+2/N, \ldots, 1-2/N, 1$. Here the summation is over the 2^N states s, $\delta(k,l)$ is the Kronecker delta and T is the temperature. The notation $\langle\ldots\rangle$ stands for the average over the random variables a_i. The limit $T \to 0$ in equation (5) ensures that only the states that minimize $E(s)$ will contribute to Z_m.

Since the average entropy $S_m(T) = -\mathrm{d}F_m/\mathrm{d}T$ of a system of Ising spins is positive at all temperatures, F_m must be a decreasing function of T, so that $\bar{E}_m = F(0) \geqslant F(T)$ for all T. Defining the annealed free energy by

$$F_m^a(T) = -T\ln\langle Z_m(T)\rangle \tag{7}$$

and using Jensen's inequality [23], $\ln\langle Z_m\rangle \geqslant \langle \ln Z_m\rangle$, yield the following inequalities

$$F_m^a(T) \leqslant F_m(T) \leqslant \bar{E}_m. \tag{8}$$

Thus, the annealed free energy calculated at any T provides a rigorous lower bound to $\bar{E}_m$ [21, 22]. Clearly, the tightest bound is given by $\bar{E}_m^a = F_m^a(T_m^*)$ where T_m^* is the temperature that maximizes $F_m^a(T)$, i.e.

$$\left.\frac{\mathrm{d}F_m^a}{\mathrm{d}T}\right|_{T_m^*} = 0. \tag{9}$$

This procedure is very useful because, in general, the annealed free energy is much easier to evaluate than the quenched one.

We now proceed with the explicit evaluation of the annealed free energy. Using the integral representations of the Dirac and Kronecker delta functions we write

$$\langle Z_m(T)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}\tilde{x}}{2\pi} \mathrm{e}^{\mathrm{i}x\tilde{x}-|x|/T} \int_{-\pi}^{\pi} \frac{\mathrm{d}\tilde{m}}{2\pi} \mathrm{e}^{\mathrm{i}Nm\tilde{m}} \prod_j \int_0^1 \mathrm{d}a_j \sum_{s_j=\pm 1} \exp[-\mathrm{i}s_j(a_j\tilde{x}+\tilde{m})]. \tag{10}$$

The integrals over x and a_j, as well as the summation over s_j, can easily be performed yielding

$$\langle Z_m(T)\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{x}}{2\pi} \frac{2T}{1+(T\tilde{x})^2} \left[\frac{\sin(\tilde{x}/2)}{\tilde{x}/2}\right]^N \int_{-\pi}^{\pi} \frac{\mathrm{d}\tilde{m}}{2\pi} \mathrm{e}^{\mathrm{i}Nm\tilde{m}} [\mathrm{e}^{\mathrm{i}\tilde{m}+\mathrm{i}\tilde{x}/2} + \mathrm{e}^{-\mathrm{i}\tilde{m}-\mathrm{i}\tilde{x}/2}]^N. \tag{11}$$

Using the binomial theorem, the integral over $\tilde{m}$ can be readily carried out. The final result is simply

$$\langle Z_m(T)\rangle = \binom{N}{n} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\pi} \frac{2T}{1+(2Ty)^2} \mathrm{e}^{NG_m(y)} \tag{12}$$

where

$$n = N\frac{1-m}{2} \tag{13}$$

$$G_m(y) = \mathrm{i}my + \ln\left(\frac{\sin y}{y}\right) \tag{14}$$

and we have made the change of variable $y = \tilde{x}/2$. In the limit of large N, the integral over y can be evaluated using the saddle-point method [24]. Since $|m| \leqslant 1$, the saddle-point is the imaginary $y_s = \mathrm{i}\zeta$, where ζ is the real solution of the equation

$$m - \coth\zeta + \frac{1}{\zeta} = 0. \tag{15}$$

Hence, the function $G_m(y_s) = G_m$, where

$$G_m = -m\zeta + \ln\frac{\sinh\zeta}{\zeta} \tag{16}$$

is real. Finally, using Stirling's formula for the binomial coefficient we rewrite equation (12) in the limit of large N as

$$\langle Z_m(T)\rangle = \frac{2}{\pi N}\sqrt{\frac{1}{(1-m^2)|G_m''|}}\frac{2T}{1-(2T\zeta)^2}\mathrm{e}^{Ng_m} \tag{17}$$

where

$$g_m = G_m - \frac{1+m}{2}\ln\frac{1+m}{2} - \frac{1-m}{2}\ln\frac{1-m}{2} \tag{18}$$

and

$$G_m'' = -1 + m^2 + \frac{2m}{\zeta}. \tag{19}$$

At this stage we can readily calculate the temperature T_m^* that maximizes the annealed free energy. In fact, equation (9) is written as

$$\ln\langle Z_m(T_m^*)\rangle + \frac{1+(2T_m^*\zeta)^2}{1-(2T_m^*\zeta)^2} = 0. \tag{20}$$

We consider first the regime where $\langle Z_m(T_m^*)\rangle$ is of order 1. In this case, equation (17) implies that T_m^* is vanishingly small, so that equation (20) reduces to $\langle Z_m(T_m^*)\rangle = \mathrm{e}^{-1}$. Inserting this result into equation (7) yields $\bar{E}_m^a = T_m^*$. Hence,

$$\bar{E}_m^a = \frac{\pi N}{4}\sqrt{(1-m^2)|G_m''|}\mathrm{e}^{-1-Ng_m} \tag{21}$$

which is consistent with the assumption that T_m^* is small for large N, provided that $g_m > 0$. Since g_m decreases monotonically with m, from $g_0 = \ln 2$ to $g_1 = -\infty$, this assumption breaks down for $|m| > 0.560$ where g_m is negative. Henceforth we will assume that $m \geqslant 0$.

It is instructive to consider in detail the case of even partitions ($m = 0$). In this case we find $\zeta = 0$, $g_0 = \ln 2$, and $G_0'' = -\frac{1}{3}$ so that

$$\langle Z_0(T)\rangle = 2^N T\frac{4\sqrt{3}}{\pi N} \tag{22}$$

and

$$\bar{E}_0^a = 2^{-N}\frac{\pi N}{4\mathrm{e}\sqrt{3}} \approx 0.1672^{-N}N. \tag{23}$$

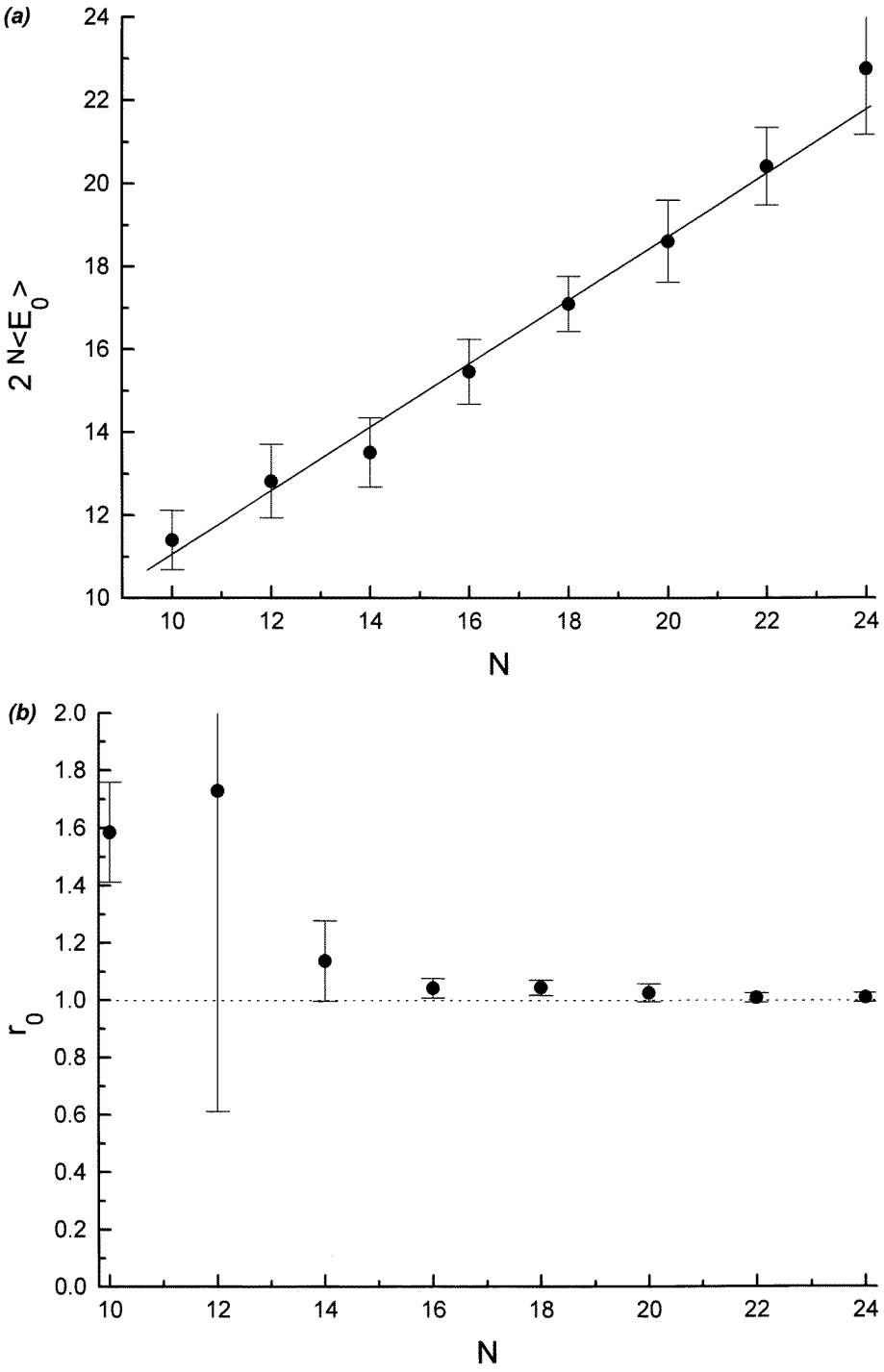

Figure 1. (*a*) Average optimal energy obtained through exhaustive search as a function of N for even partitions ($m = 0$). (*b*) Ratio between the standard deviation and the average value of the optimal energy obtained through exhaustive search as a function of N for even partitions ($m = 0$).

In figures 1(a) and (b) we present the results of numerical experiments to estimate the energy of the global minima for even partitions through the exhaustive search in the configuration space for $N \leqslant 24$. In all experiments discussed in this work, the symbols represent the averages over 10^4 realizations of the set $\{a_j\}$. The error bars are calculated by measuring the standard deviation of the average optimal energies obtained in 25 experiments, each one involving the average of 400 realizations of the set $\{a_j\}$. In these experiments we focus on the N dependence of the average optimal energy $\bar{E}_m = \langle E_m \rangle$, and of the ratio $r_m = \sqrt{\sigma_m^2}/\bar{E}_m$ where $\sigma_m^2 = \langle E_m^2 \rangle - \langle E_m \rangle^2$ is the variance of the random variable E_m. In figure 1(a) we show $\bar{E}_0$ as a function of N. The straight line shown in this figure yields the fitting $\bar{E}_0 = 0.802^{-N} N$. Hence, although the annealed bound $\bar{E}_0^a$ gives the correct scaling with N, it is about five times smaller than our numerical estimate for $\bar{E}_0$. In figure 1(b) we show the ratio r_0 as a function of N. Interestingly, this ratio tends to 1 for large N indicating then that the optimal energy E_0 is not self-averaging.

In the regime where $\langle Z_m(T_m^*) \rangle$ is of order e^N we find $\ln \langle Z_m(T_m^*) \rangle \approx N g_m$ and $T_m^* \approx 1/2\zeta$ so that

$$\bar{E}_m^a = -N \frac{g_m}{2\zeta}. \tag{24}$$

Of course, this solution is valid only for $m > 0.560$ where g_m is negative. We note that (24) gives a very poor lower bond to $\bar{E}_m$. In particular, for $m = 1$ we have $\bar{E}_1 = N/2$ while the annealed bound yields $\bar{E}_1^a = 0$. Fortunately, in the regime of extensive E_m we can use the self-averaging property of the free energy density to calculate $\bar{E}_m$ exactly for large N (see the appendix). The final result is simply

$$\bar{E}_m = \frac{N}{2}\left[\frac{(1+m)^2}{2} - 1\right] \tag{25}$$

which is valid for $m \geqslant \sqrt{2} - 1 \approx 0.414$. Thus the annealed lower bound is also very poor in the region $0.414 < m < 0.560$ since in this region $\bar{E}_m^a$ decreases exponentially with N, while $\bar{E}_m$ actually increases linearly with N.

To better appreciate the qualitative differences between the regimes of distinct scalings with N, in figure 2 we present the numerical estimates for $\bar{E}_m$ as a function of m for $N = 24$. The existence of two different regimes of scaling with N, as well as the very good agreement with the theoretical predictions for $m > 0.414$, are apparent in this figure. A noteworthy feature of our numerical estimate for $\bar{E}_m$ shown in the inset is that, in contrast to the annealed lower bound (21), the even partitions ($m = 0$) do not give the lowest energy. We have verified that this result also holds for smaller values of N. Furthermore, there seems to occur a rather abrupt transition at $m \approx 0.25$ as indicated by the large error bar and for the change of almost three orders of magnitude in $\bar{E}_m$. Although it would be very interesting to study these results more carefully for larger N, we are not aware of any efficient heuristic to solve the NPP for constrained partitions. In particular, we note that the differencing method [19] applies only to unconstrained partitions.

We turn now to the analysis of unconstrained partitions. The average partition function in this case is given by

$$\begin{aligned}\langle Z_u(T) \rangle &= \sum_m \langle Z_m(T) \rangle \\ &= 2^N \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\pi} \frac{2T}{1 + (2Ty)^2} \mathrm{e}^{N G_u(y)}\end{aligned} \tag{26}$$

where

$$G_u(y) = \ln\left[\frac{\sin(2y)}{2y}\right]. \tag{27}$$

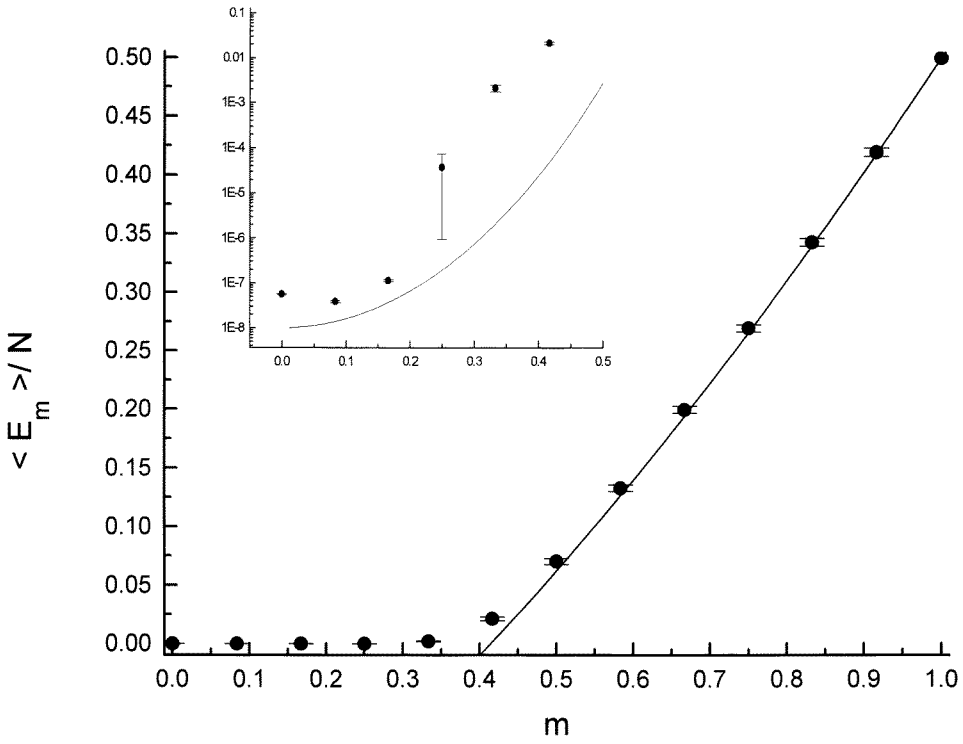

Figure 2. Average optimal energy obtained through exhaustive search as a function of m for $N = 24$. The full curve in the principal graph is the theoretical estimate (25) while the one in the inset is the annealed lower bound (21).

As before, in the limit of large N the integral over y can be carried out via a saddle-point integration. Since the saddle point is $y_s = 0$, the final result is simply

$$\langle Z_u(T)\rangle = 2^N T\sqrt{\frac{6}{\pi N}} \tag{28}$$

which yields

$$\bar{E}_u^a = 2^{-N}\sqrt{\frac{\pi N}{6e^2}} \approx 0.2662^{-N}\sqrt{N}. \tag{29}$$

It is interesting to compare this result with the average energy of a randomly chosen configuration $\boldsymbol{s}$. This quantity, which is defined by

$$\bar{E}_r = 2^{-N}\prod_i \int_0^1 \mathrm{d}a_i \sum_{s_i=\pm 1}\left|\sum_i a_i s_i\right| \tag{30}$$

is easily calculated and yields $\bar{E}_r = \sqrt{2N/3\pi}$ for large N. Moreover, comparing equations (23) and (29) we find that the lower bound for the average optimal energy of even partitions ($m = 0$), which minimizes E_m^a, is larger than that of unconstrained partitions by a factor $N^{1/2}$. The fact that these quantities do not coincide indicates that, for unconstrained partitions, m is not a self-averaging quantity, even in the large N limit, i.e. the values of m associated to the best unconstrained partitions depend on the specific realization of the set of random variables $\{a_j\}$. In figure 3(a) we present the numerical estimate for the average

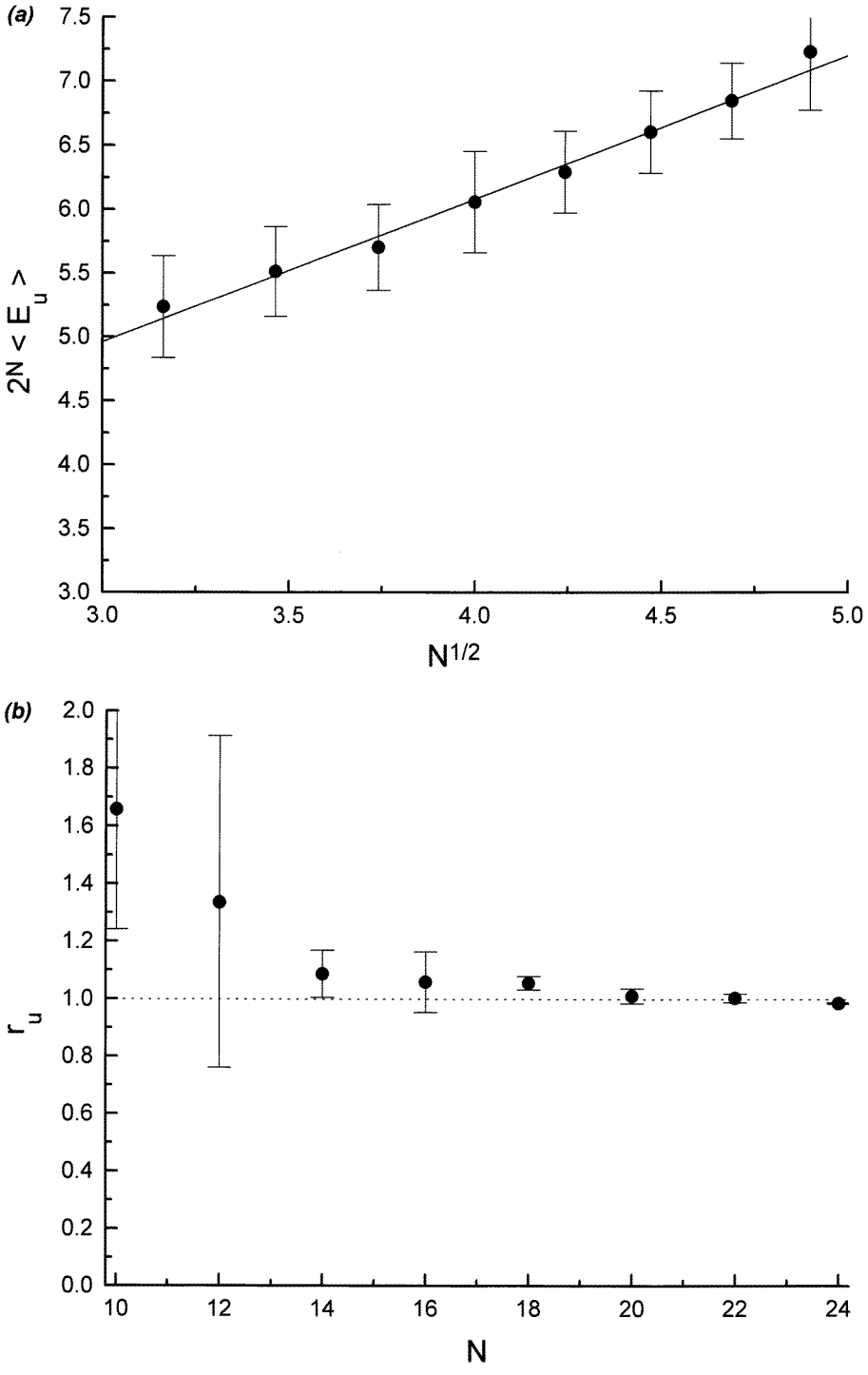

Figure 3. (*a*) Average optimal energy obtained through exhaustive search as a function of $N^{1/2}$ for unconstrained partitions. (*b*) Same as figure 1(*b*) but for unconstrained partitions.

optimal energy $\bar{E}_u = \langle E_u \rangle$ obtained through the exhaustive search for $N \leqslant 24$. The data are very well fitted by $\bar{E}_u = 1.122^{-N}\sqrt{N}$. In figure 3(*b*) we show the ratio $r_u = \sqrt{\sigma_u^2}/\bar{E}_u$ as a function of N. As before, the finding that this ratio tends to 1 for increasing N indicates that E_u is not self-averaging.

3. Average number of local minima

As mentioned before, a minimum in the 1-swap neighbourhood is a state that has lower energy than all the N states that differ from it by a single spin only [17]. In the statistical mechanics context, these states are usually termed metastable states [25]. The following analysis will be restricted to unconstrained partitions only, since for constrained partitions we would have to consider the simultaneous flip of two spins in order to satisfy the cardinality constraint. The average number local minima with energy $E = |t|$ is defined by

$$\langle \mathcal{M}(t) \rangle = \left\langle \sum_{\{s\}} \delta\left(t - \sum_j s_j a_j \right) \prod_i \Theta(|t - 2s_i a_i| - |t|) \right\rangle \tag{31}$$

where $\delta(x)$ is the Dirac delta function and $\Theta(x) = 1$ if $x \geqslant 0$ and 0 otherwise. As the calculation is straightforward we will only sketch it in the following. Using the integral representation of the delta function we obtain

$$\langle \mathcal{M}(t) \rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{t}}{2\pi} \mathrm{e}^{\mathrm{i}t\tilde{t}} \prod_j \sum_{s_j = \pm 1} \int_0^1 \mathrm{d}a_j \, \mathrm{e}^{-\mathrm{i}\tilde{t} s_j a_j} \Theta(|t - 2s_j a_j| - |t|). \tag{32}$$

Hence the integral over a_j and the summation over s_j can readily be performed, yielding

$$\langle \mathcal{M}(t) \rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{t}}{2\pi} \mathrm{e}^{\mathrm{i}t\tilde{t}} \left(\frac{\mathrm{e}^{-\mathrm{i}t\tilde{t}} - \mathrm{e}^{\mathrm{i}\tilde{t}} + \mathrm{e}^{\mathrm{i}\tilde{t}} - 1}{\mathrm{i}\tilde{t}} \right)^N \qquad \text{if } E = |t| < 1 \tag{33}$$

and

$$\langle \mathcal{M}(t) \rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{t}}{2\pi} \mathrm{e}^{\mathrm{i}t\tilde{t}} \left(\frac{\mathrm{e}^{\mathrm{i}\tilde{t}} - 1}{\mathrm{i}\tilde{t}} \right)^N = 0 \qquad \text{if } E = |t| \geqslant 1 \tag{34}$$

where we have used the interesting result that the integral in equation (34) vanishes for all N [26]. Thus, there are no local minima with $E \geqslant 1$. As usual, for large N the integral in equation (33) can be evaluated via a saddle-point integration. The final result is

$$\langle \mathcal{M}(t) \rangle = \sqrt{\frac{1}{2\pi N |H''(\xi)|}} \, \mathrm{e}^{NH(\xi)} \tag{35}$$

where

$$H(\xi) = \ln 2 + \ln\left[\frac{\sinh \xi}{\xi} - \mathrm{e}^{-t\xi/2} \frac{\sinh(t\xi/2)}{\xi} \right] \tag{36}$$

and $H''(\xi) = -\mathrm{d}^2 H(\xi)/\mathrm{d}\xi^2$. Here, ξ is the solution of

$$\frac{2}{\xi} - \frac{2\cosh \xi - \mathrm{e}^{-t\xi} t}{\sinh \xi - \mathrm{e}^{-t\xi/2} \sinh(t\xi/2)} = 0. \tag{37}$$

The function $H(\xi)$ is a monotonically decreasing function of $E = |t|$. In particular, it decreases from $\ln 2$ at $E = 0$ ($\xi = 0$) to $-\infty$ at $E = 1$ ($\xi = -\infty$). It vanishes at $E \approx 0.851$, so the average number of local minima with energy larger than that value decreases exponentially with N.

A more interesting quantity is the average number of local minima regardless of their energy values, which is defined by

$$\langle \mathcal{M} \rangle = \int_{-\infty}^{\infty} \mathrm{d}t \, \langle \mathcal{M}(t) \rangle. \tag{38}$$

From the above discussion, it is clear that only the close neighbourhood of $t = 0$ contributes to this integral, so we can expand the integrand of (33) in powers of t and $\tilde{t}$ and keep the lowest-order terms only. The final result is

$$\langle \mathcal{M} \rangle = \sqrt{\frac{24}{\pi}} \frac{2^N}{N^{3/2}} \approx 2.764 \frac{2^N}{N^{3/2}}. \tag{39}$$

It is interesting to estimate the dependence on N of the typical energy of a local minimum. This quantity, denoted by E_t, is defined by

$$E_t = \left\langle \frac{\int \mathrm{d}t \, t \mathcal{M}(t)}{\int \mathrm{d}t \, \mathcal{M}(t)} \right\rangle \tag{40}$$

which, in the annealed approximation framework [25], is approximated by

$$E_t \approx \frac{\int \mathrm{d}t \, t \langle \mathcal{M}(t) \rangle}{\langle \mathcal{M} \rangle}. \tag{41}$$

The procedure to evaluate (41) is identical to that used in the evaluation of (38) and yields

$$E_t \approx \frac{2}{N}. \tag{42}$$

We note that while equation (39) gives the exact leading-order term of the average number of local minima, equation (42) is an uncontrolled estimate for the energy of a typical minimum. These quantities can be easily estimated numerically: for each value of N, ranging from 100 to 3000, we generate 10^5 random states $\boldsymbol{s}$ and count the fraction of them that are local minima and measure their energies. We find that the numerical data are very well fitted by the equations $\langle \mathcal{M} \rangle \approx (2.81 \pm 0.02) 2^N / N^{3/2}$ and $E_t \approx (1.76 \pm 0.04)/N$, which are in quite good agreement with the theoretical predictions.

4. Conclusion

To appreciate some of the drastic features of the energy landscape associated to the NPP or, equivalently, to the random antiferromagnetic Ising model defined by the Hamiltonian (4), we compare our results with those of the SK model, which is defined by the Hamiltonian [27]

$$\mathcal{H} = -\sum_i \sum_{j>i} J_{ij} s_i s_j \tag{43}$$

where the couplings J_{ij} are Gaussian statistically independent random variables of zero mean and variance $1/N$. In this model the annealed lower bound for the ground-state energy is $E^a = -0.833N$ [21] and the number of metastable states increases as $\mathrm{e}^{0.199N}$ [28]. Hence, in the NPP there are much more local minima and the global minima are much deeper than in the SK model. These findings may explain the failure of local search techniques to produce good solutions to the NPP.

Some comments regarding the comparison of our approach with that of Karmakar *et al* [18] are in order. Those authors have derived bounds on the probability of occurrence of

the event $\mathcal{N}(E) = 0$, where $\mathcal{N}(E)$ stands for the number of states s with energy smaller than E. Interestingly, these bounds are related to the first two moments of $\mathcal{N}$:

$$1 - \langle \mathcal{N} \rangle \leqslant Pr\{\mathcal{N} = 0\} \leqslant \frac{\langle \mathcal{N}^2 \rangle - \langle \mathcal{N} \rangle^2}{\langle \mathcal{N}^2 \rangle}. \tag{44}$$

The first inequality follows trivially from the fact that $\mathcal{N} \geqslant 0$, while the second is an improvement of Chebyshev's inequality. Only unconstrained and even partitions ($m = 0$) were considered. However, as acknowledged by Karmakar *et al* [18], these bounds give no information on the average value of the difference for the best partition, except perhaps for its scaling with N. Also, we should mention that Fu [13] has actually carried out a replica analysis of the NPP for the case of even partitions $m = 0$. The aim of that analysis was to investigate the existence of a spin glass phase transition in that problem. However, since it was assumed that the free energy is extensive, the analysis missed the low-temperature phase completely and so it failed to find a phase transition. More recently, Gent and Walsh [29] presented numerical evidences for the existence of a phase transition in the NPP.

Acknowledgments

JFF thanks Pablo Moscato for useful conversations. This work was supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

Appendix

In this appendix we calculate exactly the average optimal energy in the regime where E_m scales linearly with N. Similarly to equation (10) we write the partition function defined in (6) as

$$Z_m(T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}\tilde{x}}{2\pi} \mathrm{e}^{\mathrm{i}x\tilde{x} - \beta|x|} \int_{-\pi}^{\pi} \frac{\mathrm{d}\tilde{m}}{2\pi} \mathrm{e}^{\mathrm{i}Nm\tilde{m}} \prod_j \sum_{s_j = \pm 1} \exp[-\mathrm{i}s_j(a_j\tilde{x} + \tilde{m})] \tag{45}$$

where $\beta = 1/T$ is the inverse temperature. As in the annealed approximation, the summation over s_j can easily be performed, yielding

$$\begin{aligned} Z_m(T) = N\beta^2 \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\tilde{x}}{2\pi\,\mathrm{i}} \int_{-\mathrm{i}\pi/\beta}^{\mathrm{i}\pi/\beta} \frac{\mathrm{d}\tilde{m}}{2\pi\,\mathrm{i}} \exp[-N\beta(x\tilde{x} + |x| + m\tilde{m})] \\ \times \exp\left[N \int_0^1 \mathrm{d}a\ \ln 2\cosh\beta(\tilde{x}a + \tilde{m})\right] \end{aligned} \tag{46}$$

where we have used the self-averaging property

$$\frac{1}{N}\sum_j \ln 2\cosh\beta(\tilde{x}a_j + \tilde{m}) = \int_0^1 \mathrm{d}a\ \ln 2\cosh\beta(\tilde{x}a + \tilde{m}) \tag{47}$$

which is exact for $N \to \infty$. In this limit we can carry out the integrals using the saddle-point method, and so we obtain the following equation for the average free-energy density $\bar{f}_m = \bar{F}_m/N$:

$$\bar{f}_m = x\tilde{x} + |x| + m\tilde{m} - \frac{1}{\beta}\int_0^1 \mathrm{d}a\ \ln 2\cosh\beta(\tilde{x}a + \tilde{m}). \tag{48}$$

In the zero-temperature limit ($\beta \to \infty$), the saddle-point equations yield $\tilde{x} = -1$, $\tilde{m} = (1+m)/2$ and

$$x = \frac{(1+m)^2}{4} - \frac{1}{2} \tag{49}$$

where we have assumed $x \geqslant 0$ and $m \geqslant 0$. The average optimal energy is obtained by taking the zero-temperature limit in equation (48) which yields $\bar{f}_m \to \bar{E}_m/N = |x|$.

References

[1] Anderson P W 1986 *Physica* **140A** 405
[2] Mézard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[3] Anderson P W and Fu Y 1986 *J. Phys. A: Math. Gen.* **19** 1605
[4] Mézard M and Parisi G 1986 *J. Physique* **47** 1285
[5] Cerf N J, Monvel J B, Bohigas O, Martin O C and Percus A G 1997 *J. Physique* I **7** 117
[6] Korutcheva E, Opper M and Lòpez L 1994 *J. Phys. A: Math. Gen.* **27** L645
[7] Fontanari J F 1995 *J. Phys. A: Math. Gen.* **28** 4751
[8] Inoue J 1997 *J. Phys. A: Math. Gen.* **30** 1047
[9] Kirkpatrick S and Selman B 1994 *Science* **264** 1297
[10] Monasson R and Zecchina R 1996 *Phys. Rev. Lett.* **76** 3881
[11] Hayes B 1997 *Am. Sci.* **85** 108
[12] Garey M R and Johnson D S 1979 *Computers and Intractability: A Guide to the Theory of NP-Completeness* (San Francisco, CA: Freeman)
[13] Fu Y T 1989 *Lectures in the Sciences of Complexity* ed D Stein (Reading, MA: Addison-Wesley)
[14] Shamir A 1979 *Proc. 11th Annu. ACM Symp. on Theory of Computing* p 118
[15] Kirkpatrick S, Gelatt C D and Vecchi M P 1983 *Science* **220** 671
[16] Diaz B A and Moscato P 1996 *Las Nuevas Tecnicas Heuristicas y las Redes Neuronales* ed B A Diaz (Madrid: Paraninfo)
[17] Johnson D S, Aragon C R, McGeoch L A and Schevon C 1991 *Oper. Res.* **39** 378
[18] Karmakar N, Karp R M, Lueker G S and Odlyzko A M 1986 *J. Appl. Prob.* **23** 626
[19] Karmakar N and Karp R M 1982 The differencing method of set partitioning *Report* UCB/CSD 81/113, Computer Science Division, University of California, Berkeley
[20] Ruml W, Ngo J T, Marks J and Shieber 1996 *J. Optim. Theory Appl.* **89** 251
[21] Toulouse G and Vannimenus J 1980 *Phys. Rep.* **67** 47
[22] Vannimenus J and Mézard M 1984 *J. Phys. Lett.* **45** L1145
[23] Feller W 1957 *An Introduction to Probability Theory and its Applications* vol II (New York: Wiley)
[24] Daniels H E 1954 *Ann. Math. Stat.* **25** 631
[25] Tanaka F and Edwards S F 1980 *J. Phys. F: Met. Phys.* **13** 2769
[26] Erdélyi A 1954 *Tables of Integrals Transforms* (New York: McGraw-Hill)
[27] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1972
[28] Bray A J and M A Moore 1980 *J. Phys. C: Solid State Phys.* **13** L469
[29] Gent I P and Walsh T 1996 *Proc. 12th European Conf. on Artificial Intelligence* ed W Wahlster (New York: Wiley)